

A Counterexample to a Corner-Peeling Conjecture for Subcomplexes of $\mathbb{Z}^3$

Alper Ferudun*

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Abstract

Let G be a finite connected subgraph of the grid $\mathbb{Z}^3$, not necessarily induced, and let $X(G)$ be the cube complex whose cells are the squares and 3-cubes of $\mathbb{Z}^3$ all of whose vertices and edges lie in G . A corner is a vertex that lies in exactly one maximal cell, and a corner peeling is an ordering $v_1, \dots, v_n$ of the vertices in which each v_i is a corner of the subgraph induced by $v_1, \dots, v_i$. In the problem session of the 2026 Oberwolfach workshop *Median Geometry and Applications*, V. Chepoi, from discussions with J. Chalopin and M. Kokkou, conjectured that G admits a corner peeling whenever $X(G)$ is simply connected and every connected component of every section of $X(G)$ by a coordinate plane is simply connected. We show that the conjecture is false. We give a subgraph G_{62} of $\mathbb{Z}^3$ with 62 vertices in $[0, 3]^3$ and an induced subgraph H_{73} with 73 vertices in $[0, 4]^3$ such that the cube complex and all its sections are collapsible, but there is no corner at all. Both examples are invariant under a cyclic group of order 6, and the disproof can be checked by hand; we also give explicit sequences of elementary collapses as machine-checkable certificates. A subdivision construction turns every finite subgraph into an induced subgraph whose pieces and corners are dilates of the original ones, so the conjecture for partial subgraphs and the conjecture for induced subgraphs are equivalent. Uncertified solver computations indicate that no counterexample fits into a $3 \times 3 \times 3$ box of lattice points. We also record a smaller counterexample, with 49 vertices, found in an exploratory solver search during an independent verification of this note and checked by computer. This is an unrefereed note.

1 Introduction

We write $\mathbb{Z}^3$ both for the integer lattice and for the grid graph on it, in which two points are adjacent when they differ by a unit vector $\pm e_i$. For $b \in \mathbb{Z}^3$ and $D \subseteq \{1, 2, 3\}$ let

$$C(b, D) = \{x \in \mathbb{R}^3 : b_i \leq x_i \leq b_i + 1 \ (i \in D), \ x_i = b_i \ (i \notin D)\},$$

the unit cube of dimension $|D|$ with lowest vertex b . These cubes form the standard cubical structure of $\mathbb{R}^3$. We follow the problem session of the Oberwolfach workshop *Median Geometry and Applications* (February 2026) [Che26]. There G is a “finite connected (partial) subgraph of $\mathbb{Z}^3$ ”, so G need not be an induced subgraph. Its *cube complex* $X(G)$ consists of all cubes $C(b, D)$ whose vertices and edges all belong to G ; thus G is the 1-skeleton of $X(G)$. A *corner* of $X(G)$, or of G , is a vertex of G that belongs to a unique maximal cell of $X(G)$. A *corner peeling* of G is an ordering $v_1, \dots, v_n$ of $V(G)$ such that, for every i , the vertex v_i is a corner of $G_i = G[v_1, \dots, v_i]$, the subgraph of G induced by $v_1, \dots, v_i$; note that $G_n = G$. For $i \in \{1, 2, 3\}$ and $\alpha \in \mathbb{R}$ let $\Pi_i(\alpha) = \{x \in \mathbb{R}^3 : x_i = \alpha\}$. A *section* of

*Mercury Software GmbH. Email: alper@mercurycodelab.com. GitHub: <https://github.com/AlperTheKing>.
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$X(G)$ is a nonempty intersection $X(G) \cap \Pi_i(\alpha)$, and a *piece* is a connected component of a section. It is an *integer* or a *half-integer* piece according as $\alpha \in \mathbb{Z}$ or $\alpha \in \mathbb{Z} + \frac{1}{2}$. We write x, y, z for x_1, x_2, x_3 .

The following conjecture was formulated in connection with leader election via corner peelings; see [CCK26] for leader election and metric graph theory, and [CK25] for leader election via unique sink orientations.

Conjecture 1.1 ([Che26, Problem 15]). *Let G be a finite connected subgraph of $\mathbb{Z}^3$ such that*

- (1) the cube complex $X(G)$ is simply connected, and*
- (2) each piece of $X(G)$ is simply connected.*

Then G admits a corner peeling.

It is noted in [Che26] that it suffices to require (2) for integer and half-integer pieces (see Lemma 2.2). The report also states, without proof, that the conjecture holds under an additional condition (3): for every half-integer piece H , the two half-carriers $N^\pm(H)$ of H are isometric subcomplexes of the integer pieces $H^\pm$ that contain them; and that (3) may be required for the hyperplanes of one direction only. It recalls from [CCMW22] an isometric subgraph of a 12-dimensional hypercube that has no corner, although its cube complex is contractible and all its sections are isometric subgraphs. Corners and corner peelings in related settings are also studied in [CCMW22, KM23, CC24, CM26]. The problem is record OWR-14299911-029 of the UnsolvedMath dataset [ula26].

We show that Conjecture 1.1 is false, also for induced subgraphs.

Theorem 1.2. *There are finite connected subgraphs G_{62} and H_{73} of $\mathbb{Z}^3$ with the following properties.*

- (a) G_{62} has 62 vertices and 114 edges and lies in $[0, 3]^3$. H_{73} is an induced subgraph of $\mathbb{Z}^3$ with 73 vertices and 132 edges and lies in $[0, 4]^3$.*
- (b) For $G \in \{G_{62}, H_{73}\}$, the complex $X(G)$ is collapsible, and every piece of $X(G)$ is collapsible. In particular G satisfies (1) and (2).*
- (c) Neither $X(G_{62})$ nor $X(H_{73})$ has a corner.*

Consequently neither graph admits a corner peeling, since the last vertex v_n of a corner peeling is a corner of $G_n = G$.

The graphs are defined in Sections 3 and 4 by orbit representatives under a cyclic group of order 6. Parts (a), (c), the collapsibility of all pieces and the contractibility of $X(G)$ are proved by hand from Tables 1 and 2 and Figures 1 and 2. For the collapsibility of $X(G)$ itself we give explicit sequences of elementary collapses, checked by computer.

The graph G_{62} is not induced. There are 24 pairs of adjacent lattice points in $V(G_{62})$ that are not edges of G_{62} , and the induced subgraph of $\mathbb{Z}^3$ on $V(G_{62})$ has 12 corners. This is admissible in Conjecture 1.1: its setting explicitly allows partial subgraphs, and its definition of $X(G)$ requires the edges of a cell to belong to G , a requirement that matters only for non-induced G . The example H_{73} shows that the conjecture also fails for induced subgraphs. In fact the two versions are equivalent.

Theorem 1.3. *For a finite subgraph G of $\mathbb{Z}^3$, let $G^{(2)}$ be the induced subgraph of $\mathbb{Z}^3$ on the set of points $2b + \sum_{i \in D} e_i$, one for each cell $C(b, D)$ of $X(G)$. Then $X(G^{(2)})$ is the subdivision of the dilated complex $2X(G)$ into unit cubes, so that $|X(G^{(2)})| = 2|X(G)|$. The pieces of $X(G^{(2)})$ are the sets $2P$ for the pieces P of $X(G)$, and the corners of $X(G^{(2)})$ are the points $2v$ for the corners v of $X(G)$. In particular $G^{(2)}$ is connected, satisfies (1), satisfies (2), or has no corner, if and only if G does.*

Corollary 1.4. *Conjecture 1.1 holds for all finite induced subgraphs of $\mathbb{Z}^3$ if and only if it holds for all finite subgraphs of $\mathbb{Z}^3$. The graph $G_{62}^{(2)}$ is an induced counterexample with 231 vertices in $[0, 6]^3$.*

Section 6 discusses condition (3), a 12-vertex example showing that (2) cannot be required in two directions only, and uncertified solver computations, which indicate that no counterexample fits into a $3 \times 3 \times 3$ box of lattice points. It also describes a smaller counterexample G_{49} , with 49 vertices in $[0, 3]^3$, which was found in an exploratory solver search during an independent verification of this note (see the Verification paragraph) and which we checked by computer (Remark 6.3).

2 Preliminaries

Cells and corners. All complexes below are finite subcomplexes of the cubical structure of $\mathbb{R}^3$ (or of the unit square grid of a plane). We identify a complex with its set of cells and write $|X|$ for its underlying space. The faces of $C(b, D)$ are the cubes $C(b + \sum_{i \in T} e_i, D')$ with $D' \subseteq D$ and $T \subseteq D \setminus D'$. Every point $x \in \mathbb{R}^3$ lies in the relative interior of exactly one cube, its *carrier* $C(\lfloor x \rfloor, \{i : x_i \notin \mathbb{Z}\})$, and $x \in |X|$ if and only if its carrier is a cell of X . Points of $\mathbb{Z}^3$ with one-digit coordinates are written as strings, $\mathbf{abc} = (a, b, c)$, and edges as $\mathbf{abc}-\mathbf{a}'\mathbf{b}'\mathbf{c}'$.

An *elementary collapse* removes from a complex K two cells $f \subsetneq c$ such that c is the only cell of K properly containing f . Then $\dim c = \dim f + 1$, the result is again a complex, and its underlying space is a deformation retract of $|K|$ (push c in from its free facet f). A complex is *collapsible* if a sequence of elementary collapses reduces it to a single vertex. Collapsible complexes are contractible.

Lemma 2.1. *Let G be a finite subgraph of $\mathbb{Z}^3$ and $v \in V(G)$. Let C_v be the cube with vertex v spanned by the edges of G at v , defined when (i) below holds. Then v is a corner of $X(G)$ if and only if*

- (i) $v + e_i$ and $v - e_i$ are not both neighbours of v , for each i , and
- (ii) all edges of C_v belong to G .

In that case C_v is the unique maximal cell containing v . If G is an induced subgraph of $\mathbb{Z}^3$, (ii) is equivalent to: all vertices of C_v belong to G .

Proof. Let c be a cell of $X(G)$ containing v . For each direction i of c , the edge of c at v in direction i is an edge of G . So if (i) holds, then c is a face of C_v . If also (ii) holds, then $C_v \in X(G)$, and C_v is the unique maximal cell containing v . Conversely, let v be a corner and M the maximal cell containing v . Every edge of G at v is a cell containing v and therefore lies in M . Since M has at most one edge at v in each direction, (i) holds and C_v is a face of M . As $M \in X(G)$, every edge of M at v is an edge of G , so $C_v = M$ and (ii) holds. The last claim holds because, in an induced subgraph, two vertices at distance 1 are adjacent. $\square$

Sections. The next lemma describes the sections; the half-integer case is the one used in [Che26] for the half-carriers.

Lemma 2.2. *Let $X = X(G)$, $i \in \{1, 2, 3\}$ and $k \in \mathbb{Z}$.*

- (a) $X \cap \Pi_i(k)$ is the union of the cells $C(b, D) \in X$ with $i \notin D$ and $b_i = k$, which form a subcomplex of X .
- (b) For $k < \alpha < k + 1$ we have $X \cap \Pi_i(\alpha) = X \cap \Pi_i(k + \frac{1}{2}) + (\alpha - k - \frac{1}{2})e_i$. The section $X \cap \Pi_i(k + \frac{1}{2})$ is the union of the cells $C(b, D) \cap \Pi_i(k + \frac{1}{2})$ with $C(b, D) \in X$, $i \in D$ and $b_i = k$. Its vertices correspond to the edges of G in direction i from level k to level $k + 1$, its edges to the squares of X containing two such edges, and its squares to the 3-cubes of X .

Every section is therefore a translate of a finite subcomplex of a unit square grid in a plane. Condition (2) holds as soon as all integer and half-integer pieces are simply connected.

Proof. The cube $C(b, D)$ meets $\Pi_i(\alpha)$ if and only if either $i \notin D$ and $b_i = \alpha$, or $i \in D$ and $b_i \leq \alpha \leq b_i + 1$. For $\alpha = k$ and $i \in D$, the intersection is the face $C(b, D \setminus \{i\})$ or $C(b + e_i, D \setminus \{i\})$, a cell of X in $\Pi_i(k)$; this gives (a). For $k < \alpha < k + 1$ only cells with $i \in D$ and $b_i = k$ meet $\Pi_i(\alpha)$, and $C(b, D) \cap \Pi_i(\alpha) = C(b, D \setminus \{i\}) + (\alpha - k)e_i$; this gives (b). $\square$

For a half-integer piece H in $\Pi_i(k + \frac{1}{2})$, the half-carriers of [Che26] are the translates $N^-(H) = H - \frac{1}{2}e_i \subseteq \Pi_i(k)$ and $N^+(H) = H + \frac{1}{2}e_i \subseteq \Pi_i(k + 1)$; they are subcomplexes of X .

Three topological lemmas.

Lemma 2.3. *Let K be a finite subcomplex of the unit square grid of the plane.*

- (a) *Every nonzero 2-chain of K (over any coefficient ring) has nonzero boundary. In particular $H_2(K) = 0$.*
- (b) *If K is connected and $\chi(K) = 1$, then K is collapsible.*
- (c) *If K is connected, then K is simply connected if and only if $\chi(K) = 1$, if and only if K is collapsible.*

Proof. (a) Among the squares with nonzero coefficient, choose one whose lower left corner has the largest second coordinate. Its upper edge lies in no other square with nonzero coefficient, so it appears in the boundary.

(b) We use induction on the number of squares. If K has a square, let Q be one whose lower left corner has the largest second coordinate. The upper edge e of Q lies in no other square of K , so removing e and Q is an elementary collapse. The result is connected, because the ends of e are joined by the other three edges of Q , and it has the same Euler characteristic. If K has no square, it is a connected graph with $\chi(K) = 1$, that is, a tree. Removing a leaf together with its edge is an elementary collapse, and repeating this reduces K to a vertex.

(c) Collapsible complexes are contractible, hence simply connected. If K is simply connected, then $H_1(K) = 0$, and by (a) $\chi(K) = 1 - 0 + 0 = 1$. Finally (b) applies. $\square$

Lemma 2.4. *If a finite CW complex K is the union of subcomplexes A and B such that A , B and $A \cap B$ are contractible, then K is contractible.*

Proof. Collapsing a contractible subcomplex of a CW complex to a point is a homotopy equivalence [Hat02, Prop. 0.17]. Hence $K \simeq K/(A \cap B)$, which is the wedge $A' \vee B'$ of $A' = A/(A \cap B) \simeq A$ and $B' = B/(A \cap B) \simeq B$. Collapsing A' shows $A' \vee B' \simeq B'$, which is contractible. $\square$

Lemma 2.5. *Let G be a finite nonempty subgraph of $\mathbb{Z}^3$ and $i \in \{1, 2, 3\}$, and let m and M be the minimum and the maximum of x_i on $V(G)$. If $X(G) \cap \Pi_i(\alpha)$ is contractible for every $\alpha \in \frac{1}{2}\mathbb{Z}$ with $m \leq \alpha \leq M$, then $X(G)$ is contractible.*

Proof. Let $X = X(G)$. Subdividing every cell of X into cubes of side $\frac{1}{2}$ gives a cubical complex with underlying space $|X|$ in which all sets used below are subcomplexes. For $k = m, \dots, M$ let $S_k = X \cap \{k - \frac{1}{2} \leq x_i \leq k + \frac{1}{2}\}$. The map $h_t(x) = x - t(x_i - k)e_i$, $0 \leq t \leq 1$, is a deformation retraction of S_k onto $X \cap \Pi_i(k)$. Indeed, let $x \in S_k$ lie in the cell $C(b, D)$ of X . Either $i \notin D$, and then $x_i = b_i = k$; or $i \in D$ and $b_i \in \{k - 1, k\}$, and then the segment from x to $h_1(x)$ lies in $C(b, D) \cap S_k$. Hence S_k is contractible. Moreover $X = S_m \cup \dots \cup S_M$, $S_k \cap S_{k+1} = X \cap \Pi_i(k + \frac{1}{2})$ is contractible, and $S_k \cap S_l = \emptyset$ for $|k - l| \geq 2$. So $(S_m \cup \dots \cup S_j) \cap S_{j+1} = X \cap \Pi_i(j + \frac{1}{2})$, and by Lemma 2.4 and induction on j the union $S_m \cup \dots \cup S_j$ is contractible for every $j \leq M$. $\square$

Deleting a corner. For a vertex v of G let $G - v$ be the graph obtained by deleting v and its incident edges.

Lemma 2.6. *Let G be a finite connected subgraph of $\mathbb{Z}^3$ with at least two vertices, and let v be a corner of $X(G)$. Then:*

- (a) $X(G - v)$ consists of the cells of $X(G)$ not containing v , and $X(G)$ collapses onto $X(G - v)$;
- (b) for every i and α , each piece of $X(G - v)$ in $\Pi_i(\alpha)$ is a piece of $X(G)$ or is obtained from a piece of $X(G)$ by elementary collapses, and every piece of $X(G)$ in $\Pi_i(\alpha)$ arises in this way, except possibly for one piece consisting of a single point;
- (c) $G - v$ is connected, and if G satisfies (1) or (2), then so does $G - v$.

Proof. (a) A cube of $\mathbb{Z}^3$ has all its vertices and edges in $G - v$ if and only if it is a cell of $X(G)$ not containing v . Let M be the maximal cell containing v and let D be its set of directions. Every cell containing v lies in a maximal cell, which must be M . So the cells containing v are the faces F_S of M through v , where F_S is spanned by the directions in $S \subseteq D$. As G is connected and has two vertices, v is not isolated, so $D \neq \emptyset$. Fix $d \in D$ and remove the pairs $(F_S, F_{S \cup \{d\}})$, $d \notin S$, in order of decreasing $|S|$. When such a pair is removed, every other cell containing F_S is some F_T with $T \supsetneq S$, and it was removed before. So each step is an elementary collapse.

(b) Let $\Pi = \Pi_i(\alpha)$. By Lemma 2.2 we may assume $\alpha \in \frac{1}{2}\mathbb{Z}$. The cells of the section $X(G) \cap \Pi$ that disappear are the intersections with Π of the cells of $X(G)$ containing v . If $\alpha \in \mathbb{Z}$, these are the cells of the section containing $w = v$ (if $v \in \Pi$). If $\alpha \in \mathbb{Z} + \frac{1}{2}$, a cell containing v meets Π only if it contains the edge e of G at v that crosses Π , and the cells that disappear are the cells of the section containing the point $w = e \cap \Pi$ (if e exists). In both cases they are the faces through w of the cell $M \cap \Pi$ of the section, and $M \cap \Pi$ is the unique maximal cell of the section containing w . The argument of (a) shows that they are removed by elementary collapses, unless $M \cap \Pi = \{w\}$; in that case the piece $\{w\}$ disappears.

(c) Elementary collapses preserve the homotopy type. Hence $X(G - v) \simeq X(G)$, so $X(G - v)$ is connected, and so is its 1-skeleton $G - v$. Condition (1) passes to $G - v$, and by (b) so does condition (2). $\square$

Corollary 2.7. *Let $\mathcal{G}$ be the class of all finite subgraphs of $\mathbb{Z}^3$, or the class of all finite induced subgraphs. Conjecture 1.1 holds for all members of $\mathcal{G}$ if and only if every connected $G \in \mathcal{G}$ with at least two vertices satisfying (1) and (2) has a corner. Moreover, if $G \in \mathcal{G}$ is a counterexample, then some induced subgraph $G[S]$ of G with $|S| \geq 2$ is a connected cornerless graph satisfying (1) and (2).*

Proof. A graph without corners has no corner peeling. Conversely, suppose the condition holds, and let $G \in \mathcal{G}$ be connected and satisfy (1) and (2). If G has one vertex, that vertex is a corner. Otherwise G has a corner v . The graph $G - v = G[V(G) \setminus \{v\}]$ lies in $\mathcal{G}$, is connected and satisfies (1) and (2) by Lemma 2.6. By induction it has a corner peeling $v_1, \dots, v_{n-1}$, and then $v_1, \dots, v_{n-1}, v$ is a corner peeling of G . For the last claim, delete corners one at a time as long as possible. If a single vertex remained, the reversed order of deletion would be a corner peeling of G . $\square$

3 A counterexample with 62 vertices

Let $\sigma(x, y, z) = (z, x, y)$ and $\iota(p) = (3, 3, 3) - p$. These maps are automorphisms of the grid $\mathbb{Z}^3$ that preserve the box $[0, 3]^3$. They commute, so they generate a cyclic group Γ of order 6, whose subgroup of order 3 is $\langle \sigma \rangle$.

v	orbit	edges of G_{62} at v	why v is not a corner
001	6	$+x+y+z$	spanned cube lacks the edges 002–102, 101–102
002	6	$+y+z-z$	opposite neighbours in direction z
003	6	$+x+y-z$	spanned cube lacks the edge 002–102
011	6	$+x-y+z-z$	opposite neighbours in direction z
012	6	$+x-y+z-z$	opposite neighbours in direction z
013	6	$+x-y-z$	spanned cube lacks the edge 002–102
021	6	$+x+y+z$	spanned cube lacks the edges 031–032, 131–132
022	6	$+x+y+z-z$	opposite neighbours in direction z
023	6	$+x+y-z$	spanned cube lacks the edge 123–133
111	2	$+x-x+y-y+z-z$	opposite neighbours in directions x, y, z
112	6	$+x-x+y-y+z-z$	opposite neighbours in directions x, y, z

Table 1: The 11 Γ -orbits of vertices of G_{62} , with one representative v each, the directions of the edges at v , and the condition of Lemma 2.1 that fails at v .

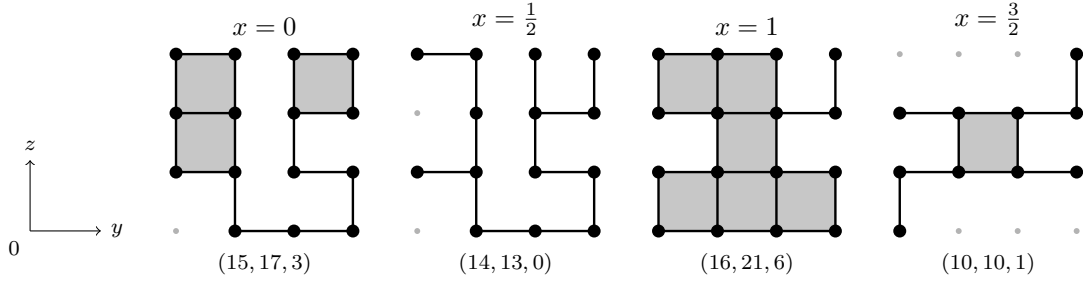


Figure 1: The sections $X(G_{62}) \cap \Pi_1(\alpha)$ for $\alpha = 0, \frac{1}{2}, 1, \frac{3}{2}$; the sections for $\alpha = 2, \frac{5}{2}, 3$ are their images under ι . The horizontal axis is y and the vertical axis is z , both from 0 to 3; gray dots are lattice points. In the integer sections, the black dots, segments and shaded squares are the vertices, edges and squares of $X(G_{62})$ in the plane. In the section $x = k + \frac{1}{2}$, a black dot at (y, z) is the edge of G_{62} from (k, y, z) to $(k+1, y, z)$, a segment is a square of $X(G_{62})$ containing two such edges, and the shaded square is the cube. Below each section are its numbers of vertices, edges and squares.

Let G_{62} be the graph whose edge set is the union of the Γ -orbits of the 19 edges

001–101, 001–011, 001–002, 002–012, 002–003, 003–103, 003–013, 011–111, 011–012, 012–112,
012–013, 013–113, 021–121, 021–031, 021–022, 022–122, 022–023, 111–211, 112–212,

and whose vertices are the endpoints of these edges.

Proposition 3.1. (a) The 19 orbits are distinct and have 6 elements each, so $|E(G_{62})| = 114$. The graph G_{62} has 62 vertices, forming the 11 Γ -orbits of Table 1.

(b) $X(G_{62})$ has 54 squares and one 3-cube, $[1, 2]^3$. The squares form the Γ -orbits of the squares with opposite corners 001, 111; 001, 012; 002, 013; 003, 113; 011, 112; 012, 113; 021, 131; 021, 122; and 111, 221. So $\chi(X(G_{62})) = 62 - 114 + 54 - 1 = 1$.

(c) Exactly 24 pairs of adjacent lattice points of $V(G_{62})$ are not edges of G_{62} . They form the Γ -orbits of 002–102, 011–021, 012–022 and 013–023. The induced subgraph of $\mathbb{Z}^3$ on $V(G_{62})$ has 12 corners, the Γ -orbits of 001 and 003.

These are finite checks. The sections in Figure 1 determine G_{62} completely, together with ι . The integer x -sections contain all edges in the directions y and z , and the half-integer x -sections record the edges in direction x , the squares containing them, and the cube.

Proof of Theorem 1.2 for G_{62} . (c) The maps in Γ preserve G_{62} , so they map corners to corners. Table 1 lists one vertex from each orbit of $V(G_{62})$, together with the directions of its edges. By Lemma 2.1 none of them is a corner: either (i) fails, or the cube C_v misses the listed edges.

(b), pieces. σ maps $\Pi_1(\alpha)$ onto $\Pi_2(\alpha)$, σ^2 maps it onto $\Pi_3(\alpha)$, and ι maps it onto $\Pi_1(3 - \alpha)$. These maps preserve $X(G_{62})$. Hence every section in a hyperplane $\Pi_i(\alpha)$ with $\alpha \in \frac{1}{2}\mathbb{Z}$ is congruent to one of the four sections of Figure 1. Sections with $\alpha \notin [0, 3]$ are empty, and sections with $\alpha \notin \frac{1}{2}\mathbb{Z}$ are translates of half-integer ones by Lemma 2.2. Each section in Figure 1 is connected, and its numbers of vertices, edges and squares are $(15, 17, 3)$, $(14, 13, 0)$, $(16, 21, 6)$ and $(10, 10, 1)$, so its Euler characteristic is 1. By Lemma 2.3 all four are collapsible. So every section of $X(G_{62})$ is a single collapsible piece.

(b), $X(G_{62})$. By the previous paragraph, the seven sections $X(G_{62}) \cap \Pi_1(\alpha)$, $\alpha = 0, \frac{1}{2}, \dots, 3$, are contractible. By Lemma 2.5, $X(G_{62})$ is contractible, and in particular simply connected. In fact $X(G_{62})$ is collapsible. An explicit sequence of 115 elementary collapses reducing it to the vertex 222 (the first collapses the cube from its bottom face) is part of the reproducibility files, and it was checked by an independent program (see the Verification paragraph).

(a) G_{62} is the 1-skeleton of the connected complex $X(G_{62})$. □

Remark 3.2. The vertex-minimising integer program of Section 6 returned, as an optimal Γ -invariant graph with 62 vertices, the image of G_{62} under the reflection $(x, y, z) \mapsto (x, z, y)$. This reflection conjugates σ to σ^{-1} and commutes with ι , so the image is again Γ -invariant; it is congruent to G_{62} and is not a new example. Its data are in the reproducibility files. A smaller counterexample, which is not Γ -invariant, is described in Remark 6.3.

4 An induced counterexample with 73 vertices

Let $\iota'(p) = (4, 4, 4) - p$ and let Γ' be the group generated by σ and ι' , again cyclic of order 6. Let V_{73} be the union of the Γ' -orbits of the 13 points

$$003, 004, 013, 014, 023, 024, 031, 032, 034, 113, 122, 123, 222,$$

and let H_{73} be the induced subgraph of $\mathbb{Z}^3$ on V_{73} . In fact H_{73} is also invariant under all permutations of the coordinates, so its symmetry group in the box $[0, 4]^3$ has order 12; we use only Γ' .

Proposition 4.1. *The point 222 is fixed by Γ' , and the other 12 orbits have 6 elements each, so $|V_{73}| = 73$. The graph H_{73} has 132 edges, in 22 orbits of size 6. $X(H_{73})$ has 60 squares, forming the Γ' -orbits of the squares with opposite corners 003, 113; 003, 104; 003, 014; 013, 123; 013, 024; 023, 124; 024, 134; 031, 132; 113, 223; and 122, 232. It has no 3-cube, so $\chi(X(H_{73})) = 73 - 132 + 60 = 1$.*

Proof of Theorem 1.2 for H_{73} . The proof is the same as for G_{62} . By Table 2 and Lemma 2.1 (induced case), no vertex is a corner. Now ι' maps $\Pi_1(\alpha)$ onto $\Pi_1(4 - \alpha)$, so every section in a hyperplane $\Pi_i(\alpha)$, $\alpha \in \frac{1}{2}\mathbb{Z}$, is congruent to one of the five sections of Figure 2. They are connected, with $(15, 18, 4)$, $(13, 12, 0)$, $(15, 18, 4)$, $(9, 8, 0)$ and $(13, 16, 4)$ vertices, edges and squares, so they are collapsible by Lemma 2.3. In particular the nine sections $X(H_{73}) \cap \Pi_1(\alpha)$, $\alpha = 0, \frac{1}{2}, \dots, 4$, are contractible, so $X(H_{73})$ is contractible by Lemma 2.5, and its 1-skeleton H_{73} is connected. An explicit sequence of 132 elementary collapses reduces $X(H_{73})$ to a vertex. □

v	orbit	edges of H_{73} at v	why v is not a corner
003	6	$+x+y+z$	spanned cube lacks the vertex 114
004	6	$+x+y-z$	spanned cube lacks the vertex 114
013	6	$+x+y-y+z$	opposite neighbours in direction y
014	6	$+y-y-z$	opposite neighbours in direction y
023	6	$+x-y+z$	spanned cube lacks the vertex 114
024	6	$+x+y-y-z$	opposite neighbours in direction y
031	6	$+x+y+z-z$	opposite neighbours in direction z
032	6	$+x+y-z$	spanned cube lacks the vertex 141
034	6	$+x+y-y$	opposite neighbours in direction y
113	6	$+x-x+y-y$	opposite neighbours in directions x, y
122	6	$+x+y+z$	spanned cube lacks the vertex 233
123	6	$+x-x+y-y+z-z$	opposite neighbours in directions x, y, z
222	1	$+x-x+y-y+z-z$	opposite neighbours in directions x, y, z

Table 2: The 13 Γ' -orbits of vertices of H_{73} , with one representative v each, the directions of the edges at v , and the condition of Lemma 2.1 that fails at v .

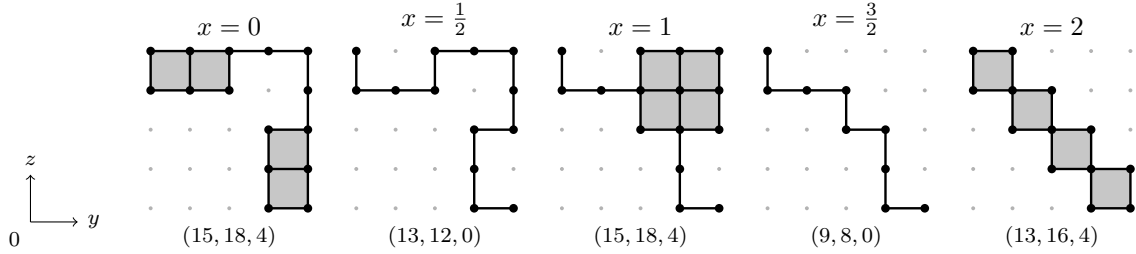


Figure 2: The sections $X(H_{73}) \cap \Pi_1(\alpha)$ for $\alpha = 0, \frac{1}{2}, 1, \frac{3}{2}, 2$; the sections for $\alpha = \frac{5}{2}, 3, \frac{7}{2}, 4$ are their images under ι' . Conventions as in Figure 1, with y and z from 0 to 4. There are no cubes, so the half-integer sections are trees.

5 Subdivision: from partial to induced subgraphs

For a cell $c = C(b, D)$ of $\mathbb{Z}^3$ put $p(c) = 2b + \sum_{i \in D} e_i$. The map $c \mapsto p(c)$ is a bijection from the cells of $\mathbb{Z}^3$ onto $\mathbb{Z}^3$, with inverse $p \mapsto C(\lfloor p/2 \rfloor, \{i : p_i \text{ odd}\})$, and $p(c)/2$ is the centre of c . If p and $q = p + e_i$ are adjacent, exactly one of p_i, q_i is odd. The cell coded by that point has the cell coded by the other point as its lower or upper facet in direction i , and every pair consisting of a cell and one of its facets arises in this way.

Proof of Theorem 1.3. Let $X = X(G)$ and $V^{(2)} = \{p(c) : c \in X\}$. A point p lies in $V^{(2)}$ if and only if $p/2 \in |X|$, since the carrier of $p/2$ is the cell coded by p .

Cells. Let $Q = C(a, S)$ be a unit cube of $\mathbb{Z}^3$. If $Q \subseteq 2|X|$, all vertices of Q lie in $V^{(2)}$, and $Q \in X(G^{(2)})$ because $G^{(2)}$ is induced. Conversely, suppose that all vertices of Q lie in $V^{(2)}$. Let q be the vertex of Q with q_i odd for all $i \in S$, and let $c = C(b, D) \in X$ be the cell coded by q ; so $D \supseteq S$. For $i \in S$ we have $[a_i, a_i + 1] \subseteq [q_i - 1, q_i + 1] = [2b_i, 2b_i + 2]$. For $i \notin S$ the i -th coordinate on Q is q_i , which lies in $[2b_i, 2b_i + 2]$ if $i \in D$ and equals $2b_i$ if $i \notin D$. Hence $Q \subseteq 2c$. So the cells of $X(G^{(2)})$ are exactly the unit cubes contained in some cube $2c$, $c \in X$. These are the cells of the subdivision of $2X$ in which every cube $2C(b, D)$ is cut into $2^{|D|}$ unit cubes. In particular $|X(G^{(2)})| = 2|X|$.

Pieces. Since $|X(G^{(2)})| = 2|X|$, we have $X(G^{(2)}) \cap \Pi_i(\beta) = 2(X \cap \Pi_i(\beta/2))$ for all i and β . So the pieces of $X(G^{(2)})$ are the sets $2P$ for the pieces P of X .

Corners. Every cell of $X(G^{(2)})$ lies in some $2M$ with M maximal in X , and it is a face of a top-dimensional unit cube of $2M$, which is again a cell of $X(G^{(2)})$. So every maximal cell of $X(G^{(2)})$ is a top-dimensional unit cube of some $2M$. Conversely, let Q be a top-dimensional unit cube of $2M$, M maximal, and let $Q' \supseteq Q$ be a cell of $X(G^{(2)})$, say $Q' \subseteq 2c'$ with $c' \in X$. Then $2c'$ meets the relative interior of $2M$, so $c' \supseteq M$ and $c' = M$. Hence $\dim Q' \leq \dim M = \dim Q$ and $Q' = Q$; so Q is maximal. Now let $p = p(c) \in V^{(2)}$, where $c = C(b, D)$ has dimension d . Then $p \in 2M$ if and only if $c \subseteq M$. If $c \subseteq M = C(b', D')$, the top-dimensional unit cubes of $2M$ are the cubes $C(a, D')$ with $a_j \in \{2b'_j, 2b'_j + 1\}$ for $j \in D'$ and $a_j = 2b'_j$ for $j \notin D'$. Such a cube contains p if and only if $a_j \leq p_j \leq a_j + 1$ for all $j \in D'$. For $j \in D$ we have $p_j = 2b'_j + 1$, and both choices of a_j work. For $j \in D' \setminus D$, $p_j \in \{2b'_j, 2b'_j + 2\}$ is an endpoint of $[2b'_j, 2b'_j + 2]$, and exactly one choice works. Hence p lies in exactly $2^d m(c)$ maximal cells of $X(G^{(2)})$, where $m(c)$ is the number of maximal cells of X containing c . So p is a corner if and only if $d = 0$ and $m(c) = 1$, that is, if and only if $p = 2v$ for a corner v of X .

Finally, G and $G^{(2)}$ are the 1-skeleta of X and $X(G^{(2)})$. So $G^{(2)}$ is connected if and only if G is. Conditions (1) and (2) are topological properties of $|X|$ and of the pieces, so they transfer by the first two parts. $\square$

Proof of Corollary 1.4. Induced subgraphs are subgraphs, which gives one direction. Conversely, if Conjecture 1.1 fails for some finite subgraph, then by Corollary 2.7 there is a connected cornerless subgraph G with at least two vertices satisfying (1) and (2). By Theorem 1.3, the induced graph $G^{(2)}$ is connected, cornerless and satisfies (1) and (2), so it is a counterexample. For $G = G_{62}$ the graph $G_{62}^{(2)}$ has one vertex for each of the $62 + 114 + 54 + 1 = 231$ cells of $X(G_{62})$, and it lies in $[0, 6]^3$. $\square$

By the proof of Theorem 1.3, $G_{62}^{(2)}$ has $2(114 + 2 \cdot 54 + 3 \cdot 1) = 450$ edges, $4 \cdot 54 + 12 \cdot 1 = 228$ squares and 8 cubes; we also checked this, and the rest of Theorem 1.3 for $G_{62}^{(2)}$, by computer.

6 Further remarks

Condition (3). If the theorem stated in [Che26] under condition (3) holds, every counterexample violates (3) in all three directions. Our examples do; we measure distances in the 1-skeleta. For G_{62} , in each direction the half-integer pieces at $\alpha = \frac{1}{2}$ and $\alpha = \frac{5}{2}$ violate (3), while the piece at $\alpha = \frac{3}{2}$ satisfies it. For instance, for the piece H at $x = \frac{1}{2}$, the points 001 and 003 have distance 2 in the integer piece $X(G_{62}) \cap \Pi_1(0)$, through 002, but distance 4 in the half-carrier $N^-(H)$, because 002 is incident to no edge in direction x . For H_{73} , the half-integer pieces at $\alpha = \frac{1}{2}$ and $\alpha = \frac{7}{2}$ violate (3) in every direction. This is only a consistency check.

Topology of the hypotheses. By the next lemma, which is also used for the computations below, (1) is equivalent to contractibility under (2).

Lemma 6.1. *Let G be a finite subgraph of $\mathbb{Z}^3$ such that every half-integer piece of $X(G)$ has trivial first homology. Then $H_2(X(G)) = H_3(X(G)) = 0$. If, in addition, $X(G)$ is connected and simply connected, then $X(G)$ is contractible and $\chi(X(G)) = 1$.*

Proof. Let $X = X(G)$. As in Lemma 2.3(a), every nonzero 3-chain of X over any coefficient ring has nonzero boundary: among the cubes with nonzero coefficient choose one whose lowest vertex has the largest third coordinate; its upper face lies in no other such cube. So $H_3(X) = 0$. Applied over $\mathbb{Z}/n$, $n \geq 2$, this shows that $H_2(X)$ is torsion-free: if z is an integral 2-cycle with $nz = \partial w$ for an integral 3-chain w , then $\partial w \equiv 0$ modulo n , hence $w = nw'$ for an integral 3-chain w' , and $z = \partial w'$. By Alexander duality [Hat02, Thm. 3.44], the rank of $H_2(X)$ is the number of bounded components of $\mathbb{R}^3 \setminus |X|$.

Suppose that U is such a component. The open set U meets the interior of some unit 3-cube Q . Then $Q \notin X$, and the interior of Q lies in U . Some edge e of Q is not in X , since otherwise all vertices and edges of Q would be in G . Let e have direction i , and let $\Pi = \Pi_i(\beta)$ be the half-integer hyperplane through its midpoint m . The cells of Q containing e are not in X , so a small neighbourhood of m in Q misses $|X|$, and hence $m \in U$. The component of $\Pi \setminus |X|$ containing m lies in U , so it is bounded. By Alexander duality in the plane, the section $X \cap \Pi$ then has nonzero first homology, and so has one of its pieces. This contradicts the hypothesis. So $H_2(X)$ is a finitely generated torsion-free group of rank 0, that is, $H_2(X) = 0$. If X is also connected and simply connected, then it is a simply connected CW complex with the homology of a point, hence contractible [Hat02, Cor. 4.33], and $\chi(X) = 1$. $\square$

Two directions. Let T_{12} be the grid graph on $[0, 2] \times [0, 1] \times [0, 1]$ with the edge 100–101 removed. Then $X(T_{12})$ consists of the eight unit squares of the boundary of the box $[0, 2] \times [0, 1] \times [0, 1]$ that do not lie in the face $y = 0$. It is a disk, and every vertex lies in at least two maximal squares, so it has no corner. All x -pieces and z -pieces are collapsible, while the pieces in $\Pi_2(\alpha)$ for $0 \leq \alpha < 1$ are hexagons. So Conjecture 1.1 fails for a 12-vertex graph if condition (2) is required only in two of the three directions. By Theorem 1.3, the induced graph $T_{12}^{(2)}$, with 39 vertices, shows the same for induced subgraphs. T_{12} violates (3) in all directions, so it says nothing about the theorem under (3).

Computations. The following reduction turns the search for a counterexample in a box into a finite problem.

Proposition 6.2. *Let B be a box of lattice points. If some counterexample to Conjecture 1.1 (respectively, some induced counterexample) has all its vertices in B , then there is a cornerless subgraph (respectively, induced subgraph) G^* of the grid graph on B with at least two vertices such that every section of $X(G^*)$ has trivial first homology and $\chi(X(G^*)) = 1$.*

Proof. Take $G^* = G[S]$ from Corollary 2.7. It is a subgraph of the grid graph on B , induced if G is. Its pieces are simply connected, so every section has trivial first homology. By Lemma 6.1, $\chi(X(G^*)) = 1$. $\square$

Two groups of programs searched for such graphs G^* with integer programming and SAT solvers, with separate encodings: the search programs with which G_{62} was found, marked (F) below, and the programs of the second referee of the Verification paragraph, marked (R), with which H_{73} was found. The author repeated some of the referee’s runs (A). There is one variable for each cell of the box, and there are constraints for the rule defining $X(G)$, for the absence of corners (Lemma 2.1), and for $\chi \geq 1$ (or χ is maximised). The condition on the first homology of the sections is expressed by *region constraints*: for a set R of unit squares in a section plane, if all edges of the boundary of R belong to the section, then so do all squares of R . By Lemma 2.3(a), these constraints hold in every section with trivial first homology, and a section with a bounded complementary region R violates the constraint for R . In most runs the region constraints are added lazily: whenever a solution has a section with a bounded complementary region R , the constraint for R is added and the solver is run again. The programs (F) impose the constraints for small regions from the start.

In the $3 \times 3 \times 3$ box all region constraints are imposed from the start. There each of the 15 section planes contains only four unit squares, and the constraints for single squares follow from the rule defining $X(G)$. The programs (F) include the constraints for all sets of squares in a 2×2 block, and the complete models (R), one for CaDiCaL and one for HiGHS, contain the constraints for all sets of at least two squares in the 15 section planes (420 clauses). The CaDiCaL model (R) for this box also breaks the 48 symmetries of the box by lex-leader constraints, so its result depends on the correctness of this symmetry breaking, which was tested separately. The runs (F) and the HiGHS run (R) use

no symmetry breaking. Table 3 lists the results. The solvers were HiGHS [HH18] through SciPy, SCIP 9.2.4 [B⁺24], and CaDiCaL [BFF⁺24] in the versions bundled with PySAT [IMMS18]: version 1.5.3 in the programs (R), and version 1.9.5, the default of the SAT search script, in those of (F). No result is certified: we have neither DRAT proofs nor exact-arithmetic certificates of optimality. Here an $a \times b \times c$ box is a translate of $\{0, \dots, a-1\} \times \{0, \dots, b-1\} \times \{0, \dots, c-1\}$.

If these results are correct, then by Proposition 6.2 no counterexample fits into a $3 \times 3 \times 3$ box. Moreover 62 is then the least number of vertices of a Γ -invariant graph in $[0, 3]^3$ with the properties of G^* , and 73 is the least number of vertices of a Γ' -invariant induced graph in $[0, 4]^3$ with these properties. The symmetric rows say nothing about non-invariant graphs. Unrestricted searches in the boxes $2 \times 2 \times n$ ($n \geq 7$), $2 \times 3 \times n$ ($n \geq 5$), $2 \times 4 \times 4$, $3 \times 3 \times 4$ and larger were not completed. A transfer-matrix program written with the search programs (F) reports that no counterexample lies in a prism $\mathbb{Z} \times \{0, 1\}^2$. It was tested, but not independently re-implemented, and we do not rely on it.

Remark 6.3 (A smaller counterexample). Let V_{49} be the set of the 49 points of $\{0, 1, 2, 3\}^3$ that lie neither in $\{0, 1\}^3$ nor in $\{2, 3\}^3 \setminus \{222\}$. The induced subgraph of $\mathbb{Z}^3$ on V_{49} has 99 edges. Let G_{49} be the graph obtained from it by deleting the 12 edges in the $\langle \sigma \rangle$ -orbits of 012–022, 022–023, 031–131 and 113–213. Then G_{49} is connected and invariant under σ , and its symmetry group in the box $[0, 3]^3$ is $\langle \sigma \rangle$, of order 3. $X(G_{49})$ has 87 edges, 39 squares and no 3-cube, so $\chi(X(G_{49})) = 49 - 87 + 39 = 1$. It has no corner, $X(G_{49})$ collapses to a vertex in 87 elementary collapses, and every section of $X(G_{49})$ is a single collapsible piece: the x -sections for $\alpha = 0, \frac{1}{2}, \dots, 3$ have $(12, 14, 3)$, $(11, 10, 0)$, $(12, 14, 3)$, $(7, 6, 0)$, $(13, 16, 4)$, $(11, 10, 0)$ and $(12, 14, 3)$ vertices, edges and squares, and σ and σ^2 map them onto the y - and z -sections. So G_{49} is a counterexample to Conjecture 1.1 with 49 vertices, and by Theorem 1.3 the graph $G_{49}^{(2)}$ is an induced counterexample with $49 + 87 + 39 = 175$ vertices, more than H_{73} . Like G_{62} and H_{73} , the graph G_{49} violates (3) in every direction, at $\alpha = \frac{1}{2}$ and $\alpha = \frac{5}{2}$. We checked these claims by computer only (see the Verification paragraph). The graph G_{49} was found by the third referee of the Verification paragraph, in exploratory SAT searches among the σ -invariant graphs in $[0, 3]^3$ with a bound on the number of vertices. These searches are uncertified and incomplete. With at most 58 vertices, the solver returned G_{49} together with a disjoint hexagon; with at most 61 vertices, it returned a σ -invariant counterexample with 59 vertices and one 3-cube; a run with at most 48 vertices did not finish.

The least number of vertices of a counterexample is unknown. It is at most 49 (Remark 6.3), and at most 73 for induced subgraphs; by Lemma 2.6, a smallest counterexample has no corner.

box	subgraphs searched	result	solvers
$3 \times 3 \times 3$	all	none	HiGHS, SCIP (F); CaDiCaL (R, A); HiGHS (R)
$2 \times 2 \times n$, $n \leq 6$	all	none	CaDiCaL (F, R)
$2 \times 3 \times 3$	all	none	CaDiCaL (F, R)
$2 \times 3 \times 4$	all	none	CaDiCaL (R)
$2 \times 4 \times 4$, $3 \times 3 \times 4$	reflection-invariant	none	HiGHS (F)
$4 \times 4 \times 4$	Γ -invariant	least $ V = 62$	HiGHS (F); CaDiCaL (R, A)
$4 \times 4 \times 4$	Γ -invariant, induced	none	CaDiCaL (R, A)
$5 \times 5 \times 5$	Γ' -invariant, induced	least $ V = 73$	CaDiCaL (R, A)

Table 3: Uncertified solver results for the graphs G^* of Proposition 6.2. “None” means that the solver reported that no such G^* exists in the box. In the rows with symmetry, only invariant graphs were searched; “reflection-invariant” refers to the point reflection of the box, and $|V|$ is the number of vertices. (F) marks runs of the search programs with which G_{62} was found, (R) runs of the separate programs of the second referee of the Verification paragraph, and (A) runs of the referee’s programs that the author repeated.

Questions. (i) What is the least number of vertices, and what is the smallest bounding box, of a counterexample to Conjecture 1.1? We know only that the least number of vertices is at most 49, and at most 73 for induced subgraphs. (ii) Our examples are collapsible, so collapsibility of $X(G)$ together with (2) does not force a corner. We do not know whether (1) and (2) imply that $X(G)$ is collapsible.

Verification. The claims were checked as follows. The referees in items 3–5 are independent, AI-assisted verification runs made for this note, not peer reviews.

1. A self-contained program by the author, `verify_paper.py` (standard library only, about one second), rebuilds G_{62} from its 19 edge-orbit representatives, H_{73} from its 13 vertex-orbit representatives and G_{49} from its description in Remark 6.3, and compares them with the data files of the search programs (F) and of the referees. It checks Propositions 3.1 and 4.1, the invariance under Γ and Γ' , the orders 6, 12 and 3 of the symmetry groups of G_{62} , H_{73} and G_{49} in their boxes, and Tables 1 and 2. It finds the corners from the definition and from Lemma 2.1, and it computes Betti numbers over $\text{GF}(2)$ and $\text{GF}(1000003)$, all sections and pieces, and condition (3). It does the same for G_{49} , for the 59-vertex example of Remark 6.3, for the mirror image of G_{62} in Remark 3.2, which it identifies as such, and for $G_{62}^{(2)}$, T_{12} and $T_{12}^{(2)}$. It also checks the statements on isometry in the Scope paragraph. It writes explicit collapse sequences for $X(G)$ (115, 132, 87 and 458 elementary collapses for G_{62} , H_{73} , G_{49} and $G_{62}^{(2)}$) and for every piece of every section. A separate checker, `check_certificates.py`, shares no code with it. It rebuilds the complexes from the edge lists and replays every collapse, checking at each step that the removed face has no other coface; it also replays the collapse sequences of the third referee for G_{49} . A randomized test, `random_tests.py`, checks Lemma 2.1, Lemma 2.6 and Theorem 1.3 on 3000 random subgraphs of small boxes, with 6913 corner deletions. Figures 1 and 2 and Tables 1 and 2 were generated from the data by `make_tikz.py`.
2. The search programs (F) come with two verifiers with separate code. They check the cell counts, the Betti numbers over $\text{GF}(2)$ and $\text{GF}(10007)$, all pieces, the graphs of integer and half-integer pieces behind Lemma 2.5, and the corners, for G_{62} , for its mirror image of Remark 3.2 and for T_{12} . A third script checks condition (3).
3. A first independent referee wrote a checker from scratch. It computes Betti numbers over $\mathbb{Q}$, $\text{GF}(2)$ and $\text{GF}(3)$, greedy collapses, the pieces and condition (3). The referee reran the scripts of (F) with byte-identical outputs and pointed out that G_{62} is not induced.
4. A second independent referee wrote a checker from scratch before reading the code of (F). It computes integer homology by Smith normal form and explicit collapses of $X(G)$ and of every piece. This referee rebuilt G_{62} from the orbit representatives, found H_{73} with a separate SAT encoding, proved the subdivision construction of Theorem 1.3 and tested it on 300 random instances, and ran the searches marked (R) in Table 3.
5. A third independent referee checked the proofs and wrote a checker from scratch, from the text of the paper, before reading any of the programs above. It confirmed the finite claims about G_{62} , H_{73} , $G_{62}^{(2)}$, T_{12} and $T_{12}^{(2)}$, including integer homology by Smith normal form and every panel of Figures 1 and 2. It tested Lemmas 2.1, 2.3, 2.5 and 2.6, Theorem 1.3 and Lemma 6.1 on 3000 random graphs, and it replayed the collapse sequences of item 1. It observed that the graph of Remark 3.2 is the mirror image of G_{62} . With a third SAT encoding, which imposes all region constraints from the start, it reproduced every row of Table 3, for the $3 \times 3 \times 3$ box again with lex-leader symmetry breaking; these runs are uncertified as well. Its exploratory searches found G_{49} , which its own checker, the checker of the first referee and the programs of item 1 confirmed.

Scope and priority. Theorem 1.2 disproves Conjecture 1.1 as stated in [Che26]. This covers partial subgraphs, the setting of [Che26], and also induced subgraphs, and by Corollary 1.4 the two versions are equivalent. Neither G_{62} nor H_{73} is an isometric subgraph of $\mathbb{Z}^3$, and most of their pieces are not isometric subgraphs of the plane grid: none of the 21 pieces of $X(G_{62})$, and 9 of the 27 pieces of $X(H_{73})$, are isometric. The conjecture as stated does not require isometry, although isometry enters the source through condition (3) and the example of [CCMW22]. We do not address the variant for isometric subgraphs. Parts (a) and (c) of Theorem 1.2, the collapsibility of all pieces and the contractibility of $X(G_{62})$ and $X(H_{73})$ are proved by hand. The collapsibility of $X(G_{62})$ and $X(H_{73})$ themselves rests on the machine-checked collapse sequences; it is not needed for the disproof. The claims about G_{49} in Remark 6.3 were checked by computer only. The statements about boxes and least sizes in Section 6, apart from the bounds given by the explicit examples, are uncertified solver results, and nothing else depends on them. We read Problem 15 in the published report [Che26]. The theorem under condition (3) is stated there without proof, and we have only checked that our examples are consistent with it. We searched for later work in the arXiv API, with author queries for Chepoi, Chalopin and Kokkou and keyword queries such as corner peeling(s) and corners of cube complexes. We also searched zbMATH, Crossref and OpenAlex (which was rate-limited in our first attempts), the publication page of V. Chepoi, and the web (five searches, September 29–30, 2026). We found no statement, proof or refutation of the conjecture outside [Che26]. The preprint [CCK26] cited there does not mention corners. The recent preprint [CM26] studies corner peelings of ample sets in Cartesian products, but not subcomplexes of $\mathbb{Z}^3$. We cannot rule out unpublished work of the proposers, and this negative search is not a proof of priority.

AI-assisted tools were used to search the problem corpus and the literature, to find the counterexamples and the arguments, to write and run the verification programs, and to help prepare this manuscript. The author has reviewed the proofs, the computations and the sources and is responsible for all claims. The note has not been peer reviewed.

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