

A Reciprocal-Block Counterexample to the Forsgård–Shapiro Coefficient-Parity Bound

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Abstract

The coefficient-parity conjecture of Forsgård and Shapiro uses the indices at which $a_k^2 - a_{k-1}a_{k+1}$ is nonnegative to bound the number of real zeros of a polynomial with positive coefficients. Katkova, Shapiro, and Vishnyakova already disproved this bound in 2024. We give an alternative explicit degree-77 polynomial with positive rational coefficients for which the selected indices change parity only once, but which has at least three distinct negative real zeros. The construction joins a degree-38 block to its reciprocal. A weighted sum-of-squares identity proves the required root crossing; all coefficients and signs can also be checked by rational arithmetic.

1 The proposed bound and the construction

For a polynomial $f(z) = \sum_{k=0}^n b_k z^k$ with $b_k > 0$, set

$$c_k = b_k^2 - b_{k-1}b_{k+1}, \quad b_{-1} = b_{n+1} = 0. \quad (1)$$

List *all* indices with $c_k \geq 0$ in increasing order as $0 = k_1 < \dots < k_m = n$, and define

$$v(f) = \#\{i : 1 \leq i < m, k_{i+1} \not\equiv k_i \pmod{2}\}.$$

Conjecture 10 in Shapiro’s problem collection [1, §7, Conjecture 10], attributed there to Forsgård and Shapiro, proposes that the number of real zeros of f is at most $v(f)$. Our example violates this inequality even when only distinct real zeros are counted, and therefore also when multiplicities are counted. Positive coefficients exclude positive real zeros.

The conjecture was already disproved by Katkova, Shapiro, and Vishnyakova [2, §5, Conjecture 3], using a degree-15 polynomial with a root of multiplicity 13 and parity count at most 11, and its higher-degree primitives. Their three strictly excluded consecutive indices persist when the repeated negative root is split into distinct negative roots, so the distinct-root interpretation also fails. This note presents a different reciprocal-block construction with parity count one and three distinct negative roots; it does not claim the first disproof of the conjecture. The 2024 reference was missed in the initial version and is credited in this revision.

Here is a compact specification of every coefficient of the example. Put

$$r = 19, \quad \varepsilon = \delta = 10^{-6}, \quad w_j = \frac{4j+3}{2} \quad (0 \leq j < r), \quad (2)$$

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and define

$$S_j = \sum_{i=0}^{j-1} \frac{1}{w_i}, \quad S = S_r = \frac{178446135735943305326}{88263547578916184475},$$

$$x_j = 1 + \delta - \frac{S_j}{S} \quad (0 \leq j \leq r). \quad (3)$$

The empty sum is zero. In particular, $x_0 = 1 + \delta > x_1 > \cdots > x_r = \delta > 0$, and

$$x_j - x_{j+1} = \frac{1}{Sw_j}. \quad (4)$$

Define the degree-38 polynomial

$$A(z) = \sum_{j=0}^r x_j^2 z^{2j} + (1 - \varepsilon) \sum_{j=0}^{r-1} x_j x_{j+1} z^{2j+1}, \quad (5)$$

and then the degree-77 polynomial

$$f(z) = z^{38} A(1/z) + z^{39} A(z). \quad (6)$$

The first term in (6) denotes the reversed polynomial, so f is an ordinary polynomial, including at $z = 0$. The two coefficient blocks occupy the disjoint ranges $0, \dots, 38$ and $39, \dots, 77$. Thus every coefficient is positive and rational.

Theorem 1. *The polynomial (6) satisfies $v(f) = 1$ and has at least three distinct negative real zeros. Consequently the bound in [1, §7, Conjecture 10] is false.*

Related obstructions for globally weighted tropicalizations are given by Forsgård, Novikov, and Shapiro [3, Theorem 11]. Their tropical indices are defined by global dominance conditions, not by the neighboring-coefficient differences (1). That result is not used as a substitute for the local-index check below. The neighboring Conjecture 9 in [1] instead uses $(k+1)b_k^2 - kb_{k-1}b_{k+1} > 0$ to select its indices; the present degree-77 calculation does not establish those weighted inequalities; the weighted conjecture was also disproved by the primitive family in [2, §5, Conjecture 2].

2 The parity count

Write $A(z) = \sum_{k=0}^{38} a_k z^k$. For $0 \leq j < 19$,

$$a_{2j+1}^2 - a_{2j}a_{2j+2} = ((1 - \varepsilon)^2 - 1)x_j^2 x_{j+1}^2 < 0. \quad (7)$$

Reversal in an even-degree block preserves the parity of an index. Therefore $c_k < 0$ for every odd k in $1, 3, \dots, 37$ in the left block of f . The right block is shifted by 39, which reverses parity, so $c_k < 0$ for every even k in $40, 42, \dots, 76$.

The two central coefficients are both $a_0 = x_0^2$; each outside neighbor is $a_1 = (1 - \varepsilon)x_0x_1$. Hence

$$c_{38} = c_{39} = x_0^4 - (1 - \varepsilon)x_0^3 x_1 > 0. \quad (8)$$

Also $c_0 = b_0^2 > 0$ and $c_{77} = b_{77}^2 > 0$. Every selected index in the left block is consequently even, every selected index in the right block is odd, and both blocks contain a selected index. Their ordered parity sequence has exactly one change:

$$v(f) = 1. \quad (9)$$

No assertion about the signs at the other indices is needed.

3 The root crossing

Set $P(t) = A(-t)$ and $Q = P'(1) + P(1)/2$.

Lemma 2. *For the block (5),*

$$Q = \frac{1}{2} \sum_{j=0}^{r-1} w_j (x_j - x_{j+1})^2 - \frac{1}{4} x_0^2 + \left(r + \frac{3}{4}\right) x_r^2 + \varepsilon \sum_{j=0}^{r-1} w_j x_j x_{j+1}. \quad (10)$$

Proof. Direct differentiation gives

$$Q = \sum_{j=0}^r \left(2j + \frac{1}{2}\right) x_j^2 - (1 - \varepsilon) \sum_{j=0}^{r-1} w_j x_j x_{j+1}.$$

Expand the squares on the right side of (10). For $1 \leq j < r$, the diagonal coefficient is $(w_{j-1} + w_j)/2 = 2j + 1/2$. At $j = 0$ it is $w_0/2 - 1/4 = 1/2$, and at $j = r$ it is $w_{r-1}/2 + r + 3/4 = 2r + 1/2$. The cross terms agree as well. $\square$

For the parameters above, (4) yields

$$\sum_{j=0}^{18} w_j (x_j - x_{j+1})^2 = \frac{1}{S}, \quad \sum_{j=0}^{18} w_j = \frac{741}{2}.$$

The fraction in (3) satisfies $S > 201/100$. Since $0 < x_j \leq 1 + 10^{-6}$, Lemma 2 gives the entirely rational estimate

$$Q < \frac{50}{201} - \frac{1}{4} + \frac{79}{4 \cdot 10^{12}} + \frac{741}{2 \cdot 10^6} (1 + 10^{-6})^2 < 0. \quad (11)$$

Both numerical comparisons in this display are exact rational inequalities, obtainable by clearing positive denominators.

For $t > 0$, the reciprocal construction gives

$$g(t) := f(-t) = t^{38} P(1/t) - t^{39} P(t). \quad (12)$$

In particular,

$$g(1) = 0, \quad g'(1) = -P(1) - 2P'(1) = -2Q > 0. \quad (13)$$

Thus $t = 1$ is a simple upward-crossing zero of g . There are points $t_- < 1 < t_+$, arbitrarily close to 1, with $g(t_-) < 0 < g(t_+)$. On the other hand $g(0) = x_r^2 > 0$, and the leading coefficient of g is $-x_r^2$, so $g(t) \rightarrow -\infty$ as $t \rightarrow +\infty$. The intermediate value theorem gives one zero in $(0, t_-)$ and another in (t_+, ∞) , in addition to the zero at 1. These yield three distinct negative zeros of f . Together with (9), this proves Theorem 1.

The degree 77 is the degree of this explicit example, not a claim of minimal degree; nor is an exact count of its real zeros asserted. This is an unrefereed manuscript. The conjecture's falsity predates this note [2]; no priority for its resolution or for this alternative construction is claimed. AI-assisted tools were used in exploratory algebra, source checking, manuscript preparation, and independent exact-arithmetic checks. The author is responsible for the statements and arguments.

A Exact-arithmetic reproduction

The following standalone Python 3 code uses only its standard library. It reconstructs all coefficients and checks the sign tests without floating-point arithmetic. Multiplication by a common positive denominator, followed by division by the greatest common divisor, gives positive primitive integer coefficients without changing the roots or the signs of (1).

```
from fractions import Fraction as F
from functools import reduce
from math import gcd, lcm

r = 19
eps = delta = F(1, 10**6)
w = [F(4*j+3, 2) for j in range(r)]
S = sum((1/q for q in w), F(0))
x = [1+delta]
for q in w:
    x.append(x[-1] - 1/(S*q))
assert x[-1] == delta and all(q > 0 for q in x)
a = []
for j in range(r):
    a.extend([x[j]**2, (1-eps)*x[j]*x[j+1]])
a.append(x[r]**2)
b = list(reversed(a)) + a
den = lcm*(q.denominator for q in b)
B = [int(den*q) for q in b]
d = reduce(gcd, B)
B = [q//d for q in B]
assert len(B) == 78 and all(q > 0 for q in B)
pad = [0] + B + [0]
c = [pad[k+1]**2-pad[k]*pad[k+2] for k in range(78)]
good = [k for k in range(78) if c[k] >= 0]
assert sum((s-t) % 2 for s,t in zip(good[1:], good[:-1])) == 1
Q = sum(((F(k)+F(1,2))*(-1)**k*q
        for k,q in enumerate(a)), F(0))
assert Q < 0
def g(t):
    value = F(0)
    for q in reversed(B):
        value = value*(-t) + q
    return value
points = [F(0), F(999,1000), F(1), F(1001,1000), F(2)]
signs = [(g(t)>0)-(g(t)<0) for t in points]
assert signs == [1, -1, 0, 1, -1]
print("degree:", len(B)-1, "good indices:", good)
print("Q:", Q, "signs:", signs)
# print(B) # Optional full primitive integer coefficient list.
```

The four nonzero signs also certify negative roots in $(-999/1000, 0)$ and $(-2, -1001/1000)$, in addition to the exact root -1 . This is an optional independent certificate, not an assumption in the proof.

References

- [1] B. Shapiro, *Problems Around Polynomials: The Good, The Bad and The Ugly...*, Arnold Mathematical Journal **1** (2015), 91–104. [doi:10.1007/s40598-015-0008-4](https://doi.org/10.1007/s40598-015-0008-4). See §7, Conjectures 9 and 10.
- [2] O. Katkova, B. Shapiro, and A. Vishnyakova, *In search of Newton-type inequalities*, Journal of Mathematical Analysis and Applications **538** (2024), 128349. [doi:10.1016/j.jmaa.2024.128349](https://doi.org/10.1016/j.jmaa.2024.128349). [arXiv:2403.12200](https://arxiv.org/abs/2403.12200), §5.
- [3] J. Forsgård, D. Novikov, and B. Shapiro, *A tropical analog of Descartes' rule of signs*, arXiv:1510.03257v1 (2015). arxiv.org/abs/1510.03257. See Theorem 11.