

A Positive-Coefficient Counterexample to the Weighted Forsgård–Shapiro Parity Bound

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Abstract

Forsgård and Shapiro proposed bounding the number of real zeros of a positive-coefficient polynomial by the parity changes among all indices at which $(k+1)a_k^2 - ka_{k-1}a_{k+1}$ is positive. Katkova, Shapiro, and Vishnyakova already disproved this bound in 2024. We give an alternative counterexample with positive rational coefficients, parity count one, and at least three distinct negative real zeros. An even-degree shift of a degree-77 reciprocal-block seed makes the undesired weighted local quantities negative. A sufficiently small, strictly log-convex factorial prefix fills every missing coefficient without introducing a parity change or destroying the three roots. The explicit witness has degree 1,000,077; no optimal-degree claim is made. Its coefficients have a compact exact formula.

1 The proposed bound

For $H(z) = \sum_{k=0}^N a_k z^k$ with every $a_k > 0$, put

$$\tilde{c}_k = (k+1)a_k^2 - ka_{k-1}a_{k+1}, \quad a_{-1} = a_{N+1} = 0. \quad (1)$$

List *all* indices with $\tilde{c}_k > 0$ in increasing order, and let $v(H)$ count parity changes between consecutive listed indices. Conjecture 9 in Shapiro’s problem collection [1, §7], attributed there to Forsgård and Shapiro, asserts that the number of real zeros is at most $v(H)$. Its strict cutoff and weighted expression differ from the neighboring unweighted Conjecture 10. The latter has a degree-77 counterexample [2]; we transfer its seed to the weighted setting, rather than claiming that an unchanged seed settles both statements.

The weighted and unweighted conjectures were already disproved by Katkova, Shapiro, and Vishnyakova [3, §5, Conjectures 2 and 3]. Their positive-coefficient primitive family violates the weighted bound in sufficiently large degree. Its strictly excluded indices persist when the repeated negative root is split into distinct negative roots, so the distinct-root interpretation also fails. Our aim is a different construction: an even shift of a reciprocal-block seed, completed by a strictly positive factorial prefix, with parity count one and three distinct negative roots. This is not the first disproof of either conjecture. The 2024 reference was missed in the initial version and is credited in this revision.

Related failures of globally weighted tropical root bounds occur in Forsgård–Novikov–Shapiro [4, Theorem 11]. Global tropical dominance indices are not the local quantities (1). The argument below checks the local signs directly.

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2 The seed and the positive prefix

We restate a compact formula for the seed. Set

$$r = 19, \quad \eta = 10^{-6}, \quad \theta = 1 - \eta, \quad w_i = \frac{4i+3}{2}, \quad S_j = \sum_{i=0}^{j-1} \frac{1}{w_i}, \quad S = S_{19},$$

and $x_j = 1 + \eta - S_j/S$ for $0 \leq j \leq 19$. The numbers x_j decrease strictly from $1 + \eta$ to η . Define

$$\begin{aligned} A(z) &= \sum_{j=0}^{19} x_j^2 z^{2j} + \theta \sum_{j=0}^{18} x_j x_{j+1} z^{2j+1}, \\ F(z) &= z^{38} A(1/z) + z^{39} A(z) = \sum_{j=0}^{77} b_j z^j. \end{aligned} \tag{2}$$

The reversed expression is an ordinary polynomial. All b_j are positive rationals and $b_0 = \eta^2 = 10^{-12}$. For $g(t) = F(-t)$, exact rational evaluation gives

$$g(0) > 0, \quad g(999/1000) < -b_0, \quad g(1001/1000) > b_0, \quad g(2) < -b_0. \tag{3}$$

The short exact-arithmetic reproduction in the appendix checks these finite inequalities directly from (2), without a root solver.

Choose the even integer $M = 10^6$ and put

$$\delta = \frac{b_0}{16M 2^{2M}}, \quad a_k = \delta \frac{k!}{(M-1)!} 2^{k^2 - (M-1)^2} \quad (0 \leq k < M). \tag{4}$$

Our polynomial is

$$H(z) = \sum_{k=0}^{M-1} a_k z^k + z^M F(z). \tag{5}$$

Thus the remaining coefficients are $a_{M+j} = b_j$, $0 \leq j \leq 77$. Every coefficient is strictly positive and rational. The tiny rational prefix is not replaced by numerical zeros.

Theorem 1. *The polynomial (5) has degree 1,000,077, satisfies $v(H) = 1$, and has at least three distinct negative real zeros. Consequently the weighted parity bound [1, Conjecture 9] is false.*

3 The exact parity count

For $1 \leq k \leq M-2$, formula (4) gives

$$a_{k-1} a_{k+1} = 4 \frac{k+1}{k} a_k^2, \quad \tilde{c}_k = -3(k+1) a_k^2 < 0.$$

At the transition, $a_{M-1} = \delta$ and $a_{M-2} = \delta 2^{3-2M}/(M-1)$, so

$$\tilde{c}_{M-1} = \delta(M\delta - b_0 2^{3-2M}) < 0,$$

because $M\delta = b_0 2^{-2M}/16$.

For $1 \leq j \leq 76$, write $c_j = b_j^2 - b_{j-1} b_{j+1}$. Then

$$\tilde{c}_{M+j} = b_j^2 + (M+j) c_j. \tag{6}$$

In the reversed left block, at each odd $j = 1, 3, \dots, 37$ there are adjacent positive numbers x_u, x_{u+1} with

$$b_j^2 = \theta^2 x_u^2 x_{u+1}^2, \quad c_j = -(1 - \theta^2) x_u^2 x_{u+1}^2.$$

The same formulas hold at each even $j = 40, 42, \dots, 76$ in the right block. Since

$$M(1 - \theta^2) = 1.999999 > 0.999998000001 = \theta^2,$$

all these undesired weighted quantities are strictly negative. The central seed quantities are strictly positive:

$$c_{38} = c_{39} = x_0^3(x_0 - \theta x_1) > 0,$$

so (6) selects both $M + 38$ and $M + 39$.

The endpoint quantities are also positive: $\tilde{c}_0 = a_0^2 > 0$ and $\tilde{c}_{M+77} = (M + 78)b_{77}^2 > 0$. The sign at M is immaterial, since M is even. Every selected index at or below $M + 38$ is therefore even; every selected index at or above $M + 39$ is odd. Both parts are nonempty, giving exactly one parity change. No local quantity of forbidden parity is zero, so changing the cutoff to nonnegative would not rescue the proposed inequality.

4 Three roots survive the prefix

Let $E(t) = \sum_{k=0}^{M-1} a_k(-t)^k$. For $0 \leq k < M$, formula (4) gives $0 < a_k \leq \delta$. Consequently, for $0 \leq t \leq 2$,

$$|E(t)| \leq \delta \sum_{k=0}^{M-1} 2^k < \delta 2^M = \frac{b_0}{16M} 2^{-M}. \quad (7)$$

Since M is even, $H(-t) = t^M g(t) + E(t)$. Each of the three nonzero arguments in (3) is at least $1/2$, so

$$|t^M g(t)| > b_0 2^{-M}.$$

This is larger than (7). Thus $H(-t)$ has signs $+, -, +, -$ at $0, 999/1000, 1001/1000, 2$, respectively; the value at zero is $a_0 > 0$. The intermediate value theorem gives three distinct roots in the three disjoint intervening intervals for t , hence three distinct negative roots of H . This proves Theorem 1, independently of the exact total root count.

The degree and coefficients were chosen for transparent strict margins, not optimized. Multiplying H by a common denominator would give an integer-coefficient witness without changing roots or local signs, but the compact rational formula avoids an unnecessary enormous expansion. No independent human refereeing, formal proof-assistant certificate, minimal degree or absolute historical priority is claimed. In particular, the conjecture's falsity predates this note [3]; priority for its resolution or this alternative construction is not claimed. AI-assisted tools were used for exploration, source comparison, exact-arithmetic checks and manuscript preparation; the author is responsible for the argument.

A Exact finite seed verification

The following Python code uses rational arithmetic only. The remaining prefix signs and root-preservation estimate are the all-size identities in the body, so no million-coefficient array or floating-point underflow is needed. The source package contains more extensive checks.

```
from fractions import Fraction as Q
r, eta, M = 19, Q(1, 10**6), 10**6
w = [Q(4*i+3, 2) for i in range(r)]
S = sum((1/u for u in w), Q(0))
x, partial = [], Q(0)
for j in range(r+1):
    x.append(1+eta-partial/S)
    if j < r: partial += 1/w[j]
A = []
for j in range(r):
    A.extend([x[j]**2, (1-eta)*x[j]*x[j+1]])
A.append(x[r]**2)
b = A[::-1]+A
assert len(b) == 78 and all(u > 0 for u in b)
assert b[0] == Q(1, 10**12)
def g(t):
    value = Q(0)
    for u in b[::-1]: value = -t*value+u
    return value
assert g(Q(999,1000)) < -b[0]
assert g(Q(1001,1000)) > b[0]
assert g(Q(2)) < -b[0]
bad = list(range(1,38,2))+list(range(40,77,2))
for j in bad:
    c = b[j]**2-b[j-1]*b[j+1]
    assert b[j]**2+(M+j)*c < 0
for j in [38,39]:
    assert b[j]**2-b[j-1]*b[j+1] > 0
assert M*(1-(1-eta)**2) > (1-eta)**2
print("Exact seed signs and weighted parity checks pass")
```

References

- [1] B. Shapiro, *Problems Around Polynomials: The Good, The Bad and The Ugly...*, Arnold Mathematical Journal **1** (2015), 91–104. [doi:10.1007/s40598-015-0008-4](https://doi.org/10.1007/s40598-015-0008-4).
- [2] A. Ferudun, *A Reciprocal-Block Counterexample to the Forsgård–Shapiro Coefficient-Parity Bound*, unrefered preprint (2026). [doi:10.5281/zenodo.23050917](https://doi.org/10.5281/zenodo.23050917).
- [3] O. Katkova, B. Shapiro, and A. Vishnyakova, *In search of Newton-type inequalities*, Journal of Mathematical Analysis and Applications **538** (2024), 128349. [doi:10.1016/j.jmaa.2024.128349](https://doi.org/10.1016/j.jmaa.2024.128349). [arXiv:2403.12200](https://arxiv.org/abs/2403.12200), §5.
- [4] J. Forsgård, D. Novikov, and B. Shapiro, *A tropical analog of Descartes’ rule of signs*, [arXiv:1510.03257](https://arxiv.org/abs/1510.03257) (2015), Theorem 11.